UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
Department of Electrical &
Computer Engineering

NE 112 *Linear algebra for nanotechnology engineering*

6.7 Bases and dimension

Douglas Wilhelm Harder, LEL, M.Math.
dwharder@uwaterloo.ca
dwharder@gmail.com

CC BY NC SA

Introduction

- In this topic, we will
  - Define a basis
  - Show how to find a basis from a collection of vectors
  - See that if the span of two bases are equal, both bases must have the same number of vectors
  - Define the dimension of a vector space or subspace
  - Discuss the benefits of having an orthonormal basis

Definition

- If a collection of vectors is linearly independent,
  it is said to form a *basis* for its span
- Given a vector space,
  a collection of vectors is a basis for that vector space if
  - The collection of vectors is linearly independent
  - The span of the vectors is the entire vector space

- Notice that a subspace can have a basis, too:
  - The collection of vectors must still be linearly independent
  - The span of those vectors must equal all vectors in that subspace and nothing else

Determining when a collection is a basis

- We have already seen how to determine if a collection of vectors is a basis:
  - Given $\mathbf{u}_1, \ldots, \mathbf{u}_n$, if these vectors are linearly independent, they form a basis for the span of these vectors
  - These vectors are linearly independent if and only if the rank of the matrix $U = \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{pmatrix}$ is equal to n
  - Of course, if these n vectors are in $\mathbf{F}^m$ and $m < n$, these n vectors can never form a basis

Finding a basis

- Also, given a collection of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$,

  we have already seen how to find a subset so that that subset is a basis for the span of those vectors
  - Again, define $U = \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{pmatrix}$ and find a row-equivalent matrix that is in row-echelon form
  - If there are no free variables associated with any column,

    that is, all columns contain a leading non-zero entry
    - Then, this collection forms a basis for their span
  - If there are free variables associated with one or more columns,

    that is, there are columns not containing a leading non-zero entry
    - Remove the vectors corresponding to those columns and the remaining vectors form a basis

Example

- Do these form a basis, and if not, find a subset that does:

$$\begin{pmatrix} -1.2 \\ 4.8 \\ 6 \\ -5.4 \end{pmatrix}, \begin{pmatrix} -2.8 \\ -1.8 \\ 4 \\ -5.6 \end{pmatrix}, \begin{pmatrix} -12 \\ 0.2 \\ 4 \\ 2.2 \end{pmatrix}, \cancel{\begin{pmatrix} -9.6 \\ -15.6 \\ -7 \\ 7.2 \end{pmatrix}}$$

- We note that

$$\begin{pmatrix} -1.2 & -2.8 & -12 & -9.6 \\ 4.8 & -1.8 & 0.2 & -15.6 \\ 6 & 4 & 4 & -7 \\ -5.4 & -5.6 & 2.2 & 7.2 \end{pmatrix} \sim \begin{pmatrix} 6 & 4 & 4 & -7 \\ 0 & -5 & -3 & -10 \\ 0 & 0 & -10 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Thus, a basis for the span is Vectors 1, 2 and 3

Example

- Do these form a basis, and if not, find a subset that does:

$$\begin{pmatrix} 2.8 \\ 4 \\ -2.4 \\ 3.6 \end{pmatrix}, \cancel{\begin{pmatrix} 2.1 \\ 3 \\ -1.8 \\ 2.7 \end{pmatrix}}, \cancel{\begin{pmatrix} 1.4 \\ 2 \\ -1.2 \\ 1.8 \end{pmatrix}}, \begin{pmatrix} -8.4 \\ -12 \\ 4.2 \\ -12.6 \end{pmatrix}$$

- We note that

$$\begin{pmatrix} 2.8 & 2.1 & 1.4 & -8.4 \\ 4 & 3 & 2 & -12 \\ -2.4 & -1.8 & -1.2 & 4.2 \\ 3.6 & 2.7 & 1.8 & -12.6 \end{pmatrix} \sim \begin{pmatrix} 4 & 3 & 2 & -12 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Thus, a basis for the span is Vectors 1 and 4

Example

- Do these form a basis, and if not, find a subset that does:

$$\begin{pmatrix} 0 \\ -0.5 \\ 5 \\ -0.5 \end{pmatrix}, \begin{pmatrix} 8.8 \\ -8.3 \\ -5 \\ 11.5 \end{pmatrix}, \begin{pmatrix} -6 \\ 1.8 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 6.4 \\ -12.9 \\ 5 \\ 7.5 \end{pmatrix}$$

- We note that

$$\begin{pmatrix} 0 & 8.8 & -6 & 6.4 \\ -0.5 & -8.3 & 1.8 & -12.9 \\ 5 & -5 & 0 & 5 \\ -0.5 & 11.5 & 0 & 7.5 \end{pmatrix} \sim \begin{pmatrix} 5 & -5 & 0 & 5 \\ 0 & 11 & 0 & 8 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix}$$

- Thus, this is a basis for the span

Example

- In $\mathbf{R}^m$, the m vectors are always a basis

$$\begin{pmatrix}1\\0\\0\\\vdots\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\\vdots\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\\vdots\\0\\0\end{pmatrix},\ldots,\begin{pmatrix}0\\0\\0\\\vdots\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\\vdots\\0\\1\end{pmatrix}$$

- After all, the rank of the corresponding matrix is m

$$\begin{pmatrix}1&0&0&\cdots&0&0\\0&1&0&\cdots&0&0\\0&0&1&\cdots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\cdots&1&0\\0&0&0&\cdots&0&1\end{pmatrix}$$

Observations

Theorem

Given two basis of a vector space that is either $\mathbf{F}^m$ or a subspace thereof,

they must have the same number of vectors.

A sketch of a proof:

Suppose $\mathbf{u}_1$ and $\mathbf{u}_2$ are a basis for a vector space, and $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ is a second basis

$$\left(\begin{array}{cc:ccc} u_{1,1} & u_{2,1} & v_{1,1} & v_{2,1} & v_{3,1} \\ u_{1,2} & u_{2,2} & v_{1,2} & v_{2,2} & v_{3,2} \\ u_{1,3} & u_{2,3} & v_{1,3} & v_{2,3} & v_{3,3} \end{array}\right) \sim \left(\begin{array}{cc:ccc} \square & ? & ? & ? & ? \\ 0 & \square & ? & ? & ? \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This must be true, as we must be able to write each of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$

Observations

However, if we performed those exact same row operations on the matrix

$$\begin{pmatrix} v_{1,1} & v_{2,1} & v_{3,1} \\ v_{1,2} & v_{2,2} & v_{3,2} \\ v_{1,3} & v_{2,3} & v_{3,3} \end{pmatrix} \sim \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ 0 & 0 & 0 \end{pmatrix}$$

No matter what these values are,

the maximum rank is 2

- Thus, they are not linearly independent

If they are a basis, however, they must be linearly independent

- This leads to a contradiction, and thus both bases must have had three vectors ∎

Observations

- Also, every different linear combination of basis vectors defines a unique vector in the span
- Identically, every vector in the span of a basis must have a unique linear combination of basis vectors that equal it
  - If $\mathbf{u}_1, \ldots, \mathbf{u}_n$ is a basis, then given an arbitrary vector $\mathbf{v}$,

    the system of linear equations defined by

    $$\alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{v}$$

    either has no solutions ($\mathbf{v}$ is not in the span)

    or it has one unique solution

Definition

Definition

The *dimension* of the vector space that is the span of n vectors is said to have a dimension of n

- We will describe such a vector space to be an n-dimensional vector space
- If the span of these vectors is a subspace,

  we will describe the subspace as being n-dimensional

Example

- For example, every vector in $\mathbf{R}^3$ can be written as a linear combination of the three linearly independent vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- Thus, the span of these vectors is $\mathbf{R}^3$ and thus these form a basis for $\mathbf{R}^3$, and the dimension of $\mathbf{R}^3$ is three
  - Thus, every basis for $\mathbf{R}^3$ must have three vectors
  - Thus, every collection of three linearly independent vectors in $\mathbf{R}^3$ is a basis for $\mathbf{R}^3$
  - This particular basis is called the *canonical basis* for $\mathbf{R}^3$

Example

- For example, the following two vectors in $\mathbf{R}^3$ are linearly independent:

$$\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}$$

- Because they are independent, they form a basis for their span
- Because there are two vectors,
  dimension of the span of these two vectors is two
  - The span is a plane in $\mathbf{R}^3$ that passes through the origin

- Because there are only two vectors in this basis,
  the span cannot be all of $\mathbf{R}^3$
  - It is a subspace of $\mathbf{R}^3$

Example

- For example, the a single non-zero $\mathbf{R}^3$ is linearly independent:

$$\begin{pmatrix} -1 \\ -6 \\ -7 \end{pmatrix}$$

- Because it is independent, it forms a basis for its span
  - That is, it is a basis for all scalar multiples of it
- Because there is one vector,
  the dimension of the span of this vectors is one
  - The span is a line in $\mathbf{R}^3$ that passes through the origin

- Because there is only one vector in this basis,
  the span cannot be all of $\mathbf{R}^3$

Bases for $\mathbf{R}^m$ and $\mathbf{C}^m$

- We have already seen that these m vectors in $\mathbf{F}^m$ are linearly independent and therefore form a basis of an m-dimensional vector space

$$\begin{pmatrix}1\\0\\0\\\vdots\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\\vdots\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\\vdots\\0\\0\end{pmatrix},\ldots,\begin{pmatrix}0\\0\\0\\\vdots\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\\vdots\\0\\1\end{pmatrix}$$

  - However, every vector in $\mathbf{F}^m$ can be written as a linear combination of these vectors; thus, the span is all of $\mathbf{F}^m$
- Thus, all bases of $\mathbf{F}^m$ must have m vectors,
  and any collection of m linearly independent vectors forms a basis
- This basis is, however, called the *canonical* basis for $\mathbf{F}^m$
- The span of any collection of n linearly independent vectors where $n < m$ is a basis for an n-dimensional subspace of $\mathbf{F}^m$

Orthogonal and orthonormal bases

Definition

If the vectors in a basis are all mutually orthogonal, that basis is described as being an *orthogonal basis*

Definition

If the vectors in an orthogonal basis are also normalized, that basis is described as being an *orthonormal basis*

Orthogonal and orthonormal bases

- Claim
  - Given a basis of n vectors $\mathbf{u}_1$, …., $\mathbf{u}_n$, if we perform the Gram Schmidt algorithm on these n vectors to get n orthonormal vectors $\hat{\mathbf{u}}_1, \ldots, \hat{\mathbf{u}}_n$, then

$$\operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\} = \operatorname{span}\{\hat{\mathbf{u}}_1, \ldots, \hat{\mathbf{u}}_n\}$$

  - Because each original vector can be written as a linear combination of the orthonormal vectors,
    and because each of the orthonormal vectors is a linear combination of the original vectors,

    their spans must overlap
  - But because they are both bases with the same number of vectors,

    these spans must be identical

Orthogonal and orthonormal bases

- The clearest benefit of an orthonormal basis is as follows:
  - If $\mathbf{v} \in \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$,

    we must solve a system of linear equations in order to find the coefficients:

$$\mathbf{v} = \alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n$$

  - If no solution exists to this system,

    then $\mathbf{v} \notin \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$
- Now, given an orthonormal basis,
  - Given any $\mathbf{v}$, $\tilde{\mathbf{v}} = \langle \hat{\mathbf{u}}_1, \mathbf{v} \rangle \hat{\mathbf{u}}_1 + \cdots + \langle \hat{\mathbf{u}}_n, \mathbf{v} \rangle \hat{\mathbf{u}}_n$ is the best approximation of $\mathbf{v}$ in the span of the orthonormal basis
  - Also, $\tilde{\mathbf{v}} = \mathbf{v}$ if and only if $\mathbf{v} \in \operatorname{span}\{\hat{\mathbf{u}}_1, \ldots, \hat{\mathbf{u}}_n\}$

Summary

- Following this topic, you now
  - Understand the idea of a basis for a vector space or subspace
  - Can take a collection of vectors and find a subset that forms a basis for the span of the original collection of vectors
  - Know that all bases for a vector space or subspace must have the same number of vectors, at least for finite dimensions
  - Are aware that orthonormal bases are better than arbitrary bases, and that it is much easier to work with orthonormal bases
  - Understand that the span of a bases equals the span of the orthonormal vectors that result from the Gram-Schmidt algorithm

References

[1] https://en.wikipedia.org/wiki/Linear_independence

[2] https://en.wikipedia.org/wiki/Linear_span

[3] https://en.wikipedia.org/wiki/Basis_(linear_algebra)

[4] https://en.wikipedia.org/wiki/Gram–Schmidt_process

Acknowledgments

None so far.

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`. Mathematical equations are prepared in MathType by Design Science, Inc.

Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

https://www.rbg.ca/

for more information.

Disclaimer

These slides are provided for the NE 112 *Linear algebra for nanotechnology engineering* course taught at the University of Waterloo. The material in it reflects the authors' best judgment in light of the information available to them at the time of preparation. Any reliance on these course slides by any party for any other purpose are the responsibility of such parties. The authors accept no responsibility for damages, if any, suffered by any party as a result of decisions made or actions based on these course slides for any other purpose than that for which it was intended.